original [illegible]

Theory of Elasticity

THIRD EDITION

S. P. Timoshenko

PROFESSOR EMERITUS OF ENGINEERING MECHANICS

J. N. Goodier

PROFESSOR OF APPLIED MECHANICS
STANFORD UNIVERSITY

1934
1970

ENGINEERING SOCIETIES MONOGRAPHS

McGraw-Hill Book Company

NEW YORK | ST. LOUIS | SAN FRANCISCO | LONDON | SYDNEY
TORONTO | MEXICO | PANAMA

chapter | 4

Two-dimensional Problems in Polar Coordinates

27 | General Equations in Polar Coordinates

In discussing stresses in circular rings and disks, curved bars of narrow rectangular cross section with a circular axis, etc., it is advantageous to use polar coordinates. The position of a point in the middle plane of a plate is then defined by the distance from the origin O (Fig. 40) and by the angle θ between r and a certain axis Ox fixed in the plane.

Let us now consider the equilibrium of a small element 1234 cut out from the plate by the radial sections 04, 02, normal to the plate, and by two cylindrical surfaces 3, 1, normal to the plate. The normal stress component in the radial direction is denoted by σ_r, the normal component in the circumferential direction by σ_θ, and the shearing-stress component by $\tau_{r\theta}$, each symbol representing stress at the point r, θ, which is the midpoint P of the element. On account of the variation of stress the values at the midpoints of the sides 1, 2, 3, 4 are not quite the same

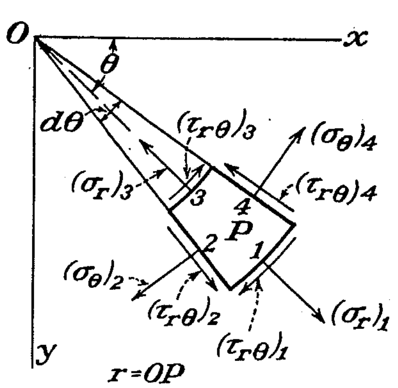

Fig. 40

as the values σ_r, σ_θ, $\tau_{r\theta}$, and are denoted by $(\sigma_r)_1$, etc., in Fig. 40. The radii of the sides 3, 1 are denoted by r_3, r_1. The radial force on the side 1 is $(\sigma_r)_1 r_1\, d\theta$ which may be written $(\sigma_r r)_1\, d\theta$, and similarly the radial force on side 3 is $-(\sigma_r r)_3\, d\theta$. The normal force on side 2 has a component along the radius through P of $-(\sigma_\theta)_2(r_1 - r_3) \sin (d\theta/2)$, which may be replaced by $-(\sigma_\theta)_2\, dr\, (d\theta/2)$. The corresponding component from side 4 is $-(\sigma_\theta)_4\, dr\, (d\theta/2)$. The shearing forces on sides 2 and 4 give $[(\tau_{r\theta})_2 - (\tau_{r\theta})_4]\, dr$.

Summing up forces in the radial direction, including body force R per unit volume in the radial direction, we obtain the equation of equilibrium

$$(\sigma_r r)_1\, d\theta - (\sigma_r r)_3\, d\theta - (\sigma_\theta)_2\, dr \frac{d\theta}{2} - (\sigma_\theta)_4\, dr \frac{d\theta}{2} + [(\tau_{r\theta})_2 - (\tau_{r\theta})_4]\, dr + Rr\, d\theta\, dr = 0$$

Dividing by $dr\, d\theta$ this becomes

$$\frac{(\sigma_r r)_1 - (\sigma_r r)_3}{dr} - \frac{1}{2}[(\sigma_\theta)_2 + (\sigma_\theta)_4] + \frac{(\tau_{r\theta})_2 - (\tau_{r\theta})_4}{d\theta} + Rr = 0$$

If the dimensions of the element are now taken smaller and smaller, to the limit zero, the first term of this equation is in the limit $\partial(\sigma_r r)/\partial r$. The second becomes σ_θ, and the third $\partial\tau_{r\theta}/\partial\theta$. The equation of equilibrium in the tangential direction may be derived in the same manner. The two equations take the final form

$$\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + R &= 0 \\ \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + S &= 0 \end{aligned} \tag{37}$$

where S is the component of body force (per unit volume) in the tangential direction (θ-increasing).

These equations take the place of Eqs. (18) when we solve two-dimensional problems by means of polar coordinates. When the body force is zero they are satisfied by putting

$$\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \end{aligned} \tag{38}$$

where ϕ is the stress function as a function of r and θ. This of course may be verified by direct substitution. A derivation of (38) is included in what follows.

Instead of deriving (37) and observing that when $R = S = 0$ they are satisfied by (38), we can consider the stress distribution in question as first given in xy components σ_x, σ_y, τ_{xy}, as in Chap. 3. We can then obtain from these the polar components σ_r, σ_θ, $\tau_{r\theta}$. From (13) we have (identifying α with θ)

$$\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \\
\tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned} \tag{a}$$

We can similarly express σ_x, σ_y, τ_{xy} in terms of σ_r, σ_θ, $\tau_{r\theta}$ by the relations (see Prob. 1, page 144)

$$\begin{aligned}
\sigma_x &= \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta \\
\sigma_y &= \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta + 2\tau_{r\theta}\sin\theta\cos\theta \\
\tau_{xy} &= (\sigma_r - \sigma_\theta)\sin\theta\cos\theta + \tau_{r\theta}(\cos^2\theta - \sin^2\theta)
\end{aligned} \tag{b}$$

To obtain (38) we consider next the relations between derivatives in the two coordinate systems. First we have

$$r^2 = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x}$$

which yield

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta \\
\frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin\theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos\theta}{r}
\end{aligned}$$

Thus for any function $f(x,y)$, in polar coordinates $f(r\cos\theta, r\sin\theta)$, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \tag{c}$$

To get $\partial^2 f/\partial x^2$ we repeat the operation indicated in the last member of (c). Then

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2 f}{\partial r^2} - \cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) \\
&\qquad - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial r}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)
\end{aligned}$$

With a little rearrangement, this takes the form

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \sin^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right) - 2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) \tag{d}$$

Similarly, we find

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta \frac{\partial^2 f}{\partial r^2} + \cos^2\theta \left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right) + 2\sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) \quad (e)$$

$$-\frac{\partial^2 f}{\partial x\,\partial y} = \sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} - \frac{\partial^2 f}{\partial r^2}\right) - (\cos^2\theta - \sin^2\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) \quad (f)$$

When we take for f the stress-function $\phi(x,y)$ as in (29)—but with $\rho g = 0$—the derivatives on the left-hand sides of (d), (e), and (f) become σ_y, σ_x, τ_{xy}, respectively. The expressions on the right-hand sides of (d), (e), and (f) can therefore be substituted for these stress components in the right-hand sides of (a). It is easily verified that the results reduce to (38).

To convert the differential equation (a), page 35, to polar form, we first add (d) and (e) above to obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f \quad (g)$$

showing that the operator on the right is the polar equivalent of the laplacian operator on the left. Next, we find by addition of the first two of equations (b)

$$\sigma_x + \sigma_y = \sigma_r + \sigma_\theta \quad (h)$$

For zero body force we have, as on page 30,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \quad (i)$$

In view of (i), (h), and (g), this becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \quad (39)$$

From various solutions of this partial differential equation we obtain solutions of two-dimensional problems in polar coordinates for various boundary conditions. Several examples of such problems will be discussed in this chapter.

28 | Stress Distribution Symmetrical about an Axis

When the stress function depends on r only, the equation of compatibility (39) becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0 \quad (40)$$

This is an ordinary differential equation, which can be reduced to a linear differential equation with constant coefficients by introducing a new variable t such that $r = e^t$. In this manner the general solution of Eq. (40) can be easily obtained. This solution has four constants of integration, which must be determined from the boundary conditions. By substitution it can be checked that

$$\phi = A \log r + Br^2 \log r + Cr^2 + D \tag{41}$$

is the general solution. The solutions of a group of problems of symmetrical stress distribution[1] with no body forces can be obtained from this. The corresponding stress components from Eqs. (38) are

$$\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} = \frac{A}{r^2} + B(1 + 2 \log r) + 2C \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2 \log r) + 2C \\ \tau_{r\theta} &= 0 \end{aligned} \tag{42}$$

If there is no hole at the origin of coordinates, constants A and B vanish, since otherwise the stress components (42) become infinite when $r = 0$. Hence, for a plate without a hole at the origin and with no body forces, only one case of stress distribution symmetrical with respect to the axis may exist, namely, that when $\sigma_r = \sigma_\theta =$ constant and the plate is in a condition of uniform tension or uniform compression in all directions in its plane.

If there is a hole at the origin, other solutions than uniform tension or compression can be derived from expressions (42). Taking B as zero,[2] for instance, Eqs. (42) become

$$\begin{aligned} \sigma_r &= \frac{A}{r^2} + 2C \\ \sigma_\theta &= -\frac{A}{r^2} + 2C \end{aligned} \tag{43}$$

This solution may be adapted to represent the stress distribution in a hollow cylinder submitted to uniform pressure on the inner and outer surfaces[3] (Fig. 41). Let a and b denote the inner and outer radii of the cylinder, and p_i and p_o the uniform internal and external pressures. Then the boundary conditions are

$$(\sigma_r)_{r=a} = -p_i \qquad (\sigma_r)_{r=b} = -p_o \tag{a}$$

[1] The stress function independent of θ does not give all stress distributions independent of θ. The function of the form $A\theta$ as in (q) on p. 126 illustrates this.

[2] Proof that B must be zero requires consideration of displacements. See p. 78.

[3] The solution of this problem is due to Lamé, "Leçons sur la théorie . . . de l'élasticité," Gauthier-Villars, Paris, 1852.

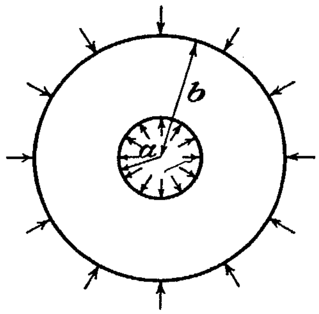

Fig. 41

Substituting in the first of Eqs. (43), we obtain the following equations to determine A and C:

$$\frac{A}{a^2} + 2C = -p_i$$

$$\frac{A}{b^2} + 2C = -p_o$$

from which

$$A = \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}$$

$$2C = \frac{p_ia^2 - p_ob^2}{b^2 - a^2}$$

Substituting these in Eqs. (43) the following expressions for the stress components are obtained: $p_o = 0$ $\quad \frac{-a^2b^2p_i + p_ia^2r^2}{(b^2-a^2)r^2} = p_ia^2(b^2+1)$

$$\begin{aligned} \sigma_r &= \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2} \\ \sigma_\theta &= -\frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2} \end{aligned} \tag{44}$$

The radial displacement u is easily found since here $\epsilon_\theta = u/r$, and for plane stress

$$E\epsilon_\theta = \sigma_\theta - \nu\sigma_r$$

It is interesting to note that the sum $\sigma_r + \sigma_\theta$ is constant through the thickness of the wall of the cylinder. Hence the stresses σ_r and σ_θ produce a uniform extension or contraction in the direction of the axis of the cylinder, and cross sections perpendicular to this axis remain plane. Hence the deformation produced by the stresses (44) in an element of the cylinder cut out by two adjacent cross sections does not interfere with the deformation of the neighboring elements, and it is justifiable to consider the element in the condition of plane stress as we did in the above discussion.

In the particular case when $p_o = 0$ and the cylinder is submitted to internal pressure only, Eqs. (44) give

$$\sigma_r = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$
$$\sigma_\theta = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{45}$$

These equations show that σ_r is always a compressive stress and σ_θ a tensile stress. The latter is greatest at the inner surface of the cylinder, where

$$(\sigma_\theta)_{\max} = \frac{p_i(a^2 + b^2)}{b^2 - a^2} \tag{46}$$

$(\sigma_\theta)_{\max}$ is always numerically greater than the internal pressure and approaches this quantity as b increases, so that it can never be reduced below p_i, however much material is added on the outside. Various applications of Eqs. (45) and (46) in machine design are usually discussed in elementary books on the strength of materials.[1]

The corresponding problem for a cylinder with an eccentric bore was solved by G. B. Jeffery.[2] If the radius of the bore is a and that of the external surface b, and if the distance between their centers is e, the maximum stress, when the cylinder is under an internal pressure p_i, is the tangential stress at the internal surface at the thinnest part, if $e < \frac{1}{2}a$, and is of the magnitude

$$\sigma = p_i\left[\frac{2b^2(b^2 + a^2 - 2ae - e^2)}{(a^2 + b^2)(b^2 - a^2 - 2ae - e^2)} - 1\right]$$

If $e = 0$, this coincides with Eq. (46).

29 | Pure Bending of Curved Bars

Let us consider a curved bar with a constant narrow rectangular cross section[3] and a circular axis bent in the plane of curvature by couples M applied at the ends (Fig. 42). The bending moment in this case is constant along the length of the bar and it is natural to expect that the stress distribution is the same in all radial cross sections, and that the solution of the problem can therefore be obtained by using expression (41).

[1] See, for instance, S. Timoshenko, "Strength of Materials," 3d ed., vol. 2, chap. 6, D. Van Nostrand Company, Inc., Princeton, N.J., 1956.

[2] *Trans. Roy. Soc. (London)*, ser. A, vol. 221, p. 265, 1921. See also *Brit. Assoc. Advan. Sci. Rept.*, 1921. A complete solution by a different method is given in Art. 66 of the present book.

[3] From the general discussion of the two-dimensional problem, Art. 16, it follows that the solution obtained below for the stress holds also for plane strain.

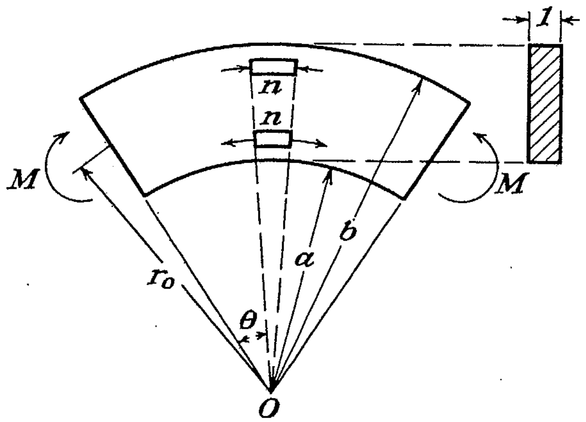

Fig. 42

Denoting by a and b the inner and the outer radii of the boundary and taking the width of the rectangular cross section as unity, the boundary conditions are

$$
\begin{aligned}
&(1) \qquad & \sigma_r &= 0 \qquad && \text{for } r = a \text{ and } r = b \\
&(2) \qquad & \int_a^b \sigma_\theta \, dr &= 0 \qquad && \int_a^b \sigma_\theta r \, dr = -M \qquad\qquad (a) \\
&(3) \qquad & \tau_{r\theta} &= 0 \qquad && \text{at the boundary}
\end{aligned}
$$

Condition (1) means that the convex and concave boundaries of the bar are free from normal forces; condition (2) indicates that the normal stresses at the ends give rise to the couple M only, and condition (3) indicates that there are no tangential forces applied at the boundary. Using the first of Eqs. (42) with (1) of the boundary conditions (a) we obtain

$$
\begin{aligned}
\frac{A}{a^2} + B(1 + 2 \log a) + 2C &= 0 \\
\frac{A}{b^2} + B(1 + 2 \log b) + 2C &= 0
\end{aligned}
\qquad (b)
$$

Condition (2) in (a) is now necessarily satisfied. The use of a stress function guarantees equilibrium. A nonzero force-resultant on each end would violate equilibrium. To have the bending couple equal to M, the condition

$$
\int_a^b \sigma_\theta r \, dr = \int_a^b \frac{\partial^2 \phi}{\partial r^2} r \, dr = -M \qquad (d)
$$

must be fulfilled. We have

$$
\int_a^b \frac{\partial^2 \phi}{\partial r^2} r \, dr = \left| \frac{\partial \phi}{\partial r} r \right|_a^b - \int_a^b \frac{\partial \phi}{\partial r} \, dr = \left| \frac{\partial \phi}{\partial r} r \right|_a^b - |\phi|_a^b
$$

and noting that on account of (*b*),

$$\left| \frac{\partial \phi}{\partial r} r \right|_a^b = 0$$

we find from (*d*),

$$|\phi|_a^b = M$$

or substituting expression (41) for ϕ,

$$A \log \frac{b}{a} + B(b^2 \log b - a^2 \log a) + C(b^2 - a^2) = M \tag{e}$$

This equation, together with the two Eqs. (*b*), completely determines the constants A, B, C, and we find

$$\begin{aligned} A &= -\frac{4M}{N} a^2b^2 \log \frac{b}{a} \qquad B = -\frac{2M}{N}(b^2 - a^2) \\ C &= \frac{M}{N}[b^2 - a^2 + 2(b^2 \log b - a^2 \log a)] \end{aligned} \tag{f}$$

where for simplicity we have put

$$N = (b^2 - a^2)^2 - 4a^2b^2 \left(\log \frac{b}{a}\right)^2 \tag{g}$$

Substituting the values (*f*) of the constants into the expressions (42) for the stress components, we find

$$\begin{aligned} \sigma_r &= -\frac{4M}{N}\left(\frac{a^2b^2}{r^2} \log \frac{b}{a} + b^2 \log \frac{r}{b} + a^2 \log \frac{a}{r}\right) \\ \sigma_\theta &= -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2} \log \frac{b}{a} + b^2 \log \frac{r}{b} + a^2 \log \frac{a}{r} + b^2 - a^2\right) \\ \tau_{r\theta} &= 0 \end{aligned} \tag{47}$$

This gives the stress distribution satisfying all the boundary conditions[1] (*a*) for pure bending and represents the exact solution of the problem, provided the distribution of the normal forces at the ends is that given by the second of Eqs. (47). If the forces giving the bending couple M are distributed over the ends of the bar in some other manner, the stress distribution at the ends will be different from that of the solution (47). But, as Saint-Venant's principle suggests, the deviations from solution (47) may be negligible away from the ends, say at distances greater than the depth of the bar. This is illustrated by Fig. 102.

[1] This solution is due to H. Golovin, *Trans. Inst. Tech.*, St. Petersburg, 1881. The paper, published in Russian, remained unknown in other countries, and the same problem was solved later by M. C. Ribière (*Compt. Rend.*, vol. 108, 1889, and vol. 132, 1901) and by L. Prandtl. See A. Föppl, "Vorlesungen uber Technische Mechanik," vol. 5, p. 72, 1907; also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

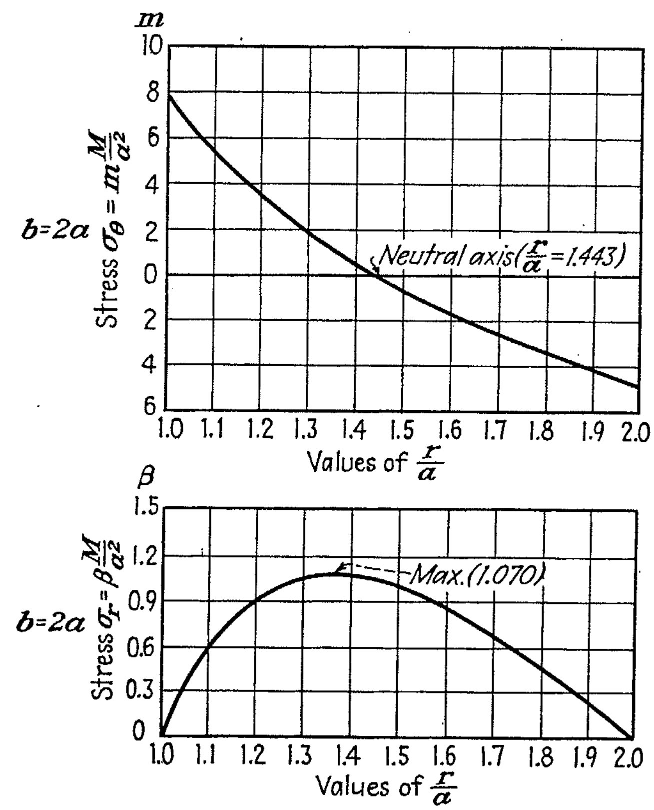

Fig. 43

From the first of Eqs. (47) it can be shown that the stress σ_r is always positive for the direction of bending shown in Fig. 42. The same can be concluded at once from the direction of stresses σ_θ acting on the elements $n - n$ in Fig. 42. The corresponding tangential forces give resultants in the radial direction tending to separate longitudinal fibers and producing tensile stress in the radial direction. This stress increases toward the neutral surface and becomes a maximum near this surface. This maximum is always much smaller than $(\sigma_\theta)_{\max}$. For instance, for $b/a = 1.3$, $(\sigma_r)_{\max} = 0.060(\sigma_\theta)_{\max}$; for $b/a = 2$, $(\sigma_r)_{\max} = 0.138(\sigma_\theta)_{\max}$; for $b/a = 3$, $(\sigma_r)_{\max} = 0.193(\sigma_\theta)_{\max}$. In Fig. 43 the distribution of σ_θ and σ_r for $b/a = 2$ is given. From this figure we see that the point of maximum stress σ_r is somewhat displaced from the neutral axis in the direction of the center of curvature.

30 | Strain Components in Polar Coordinates

In considering the displacement in polar coordinates let us denote by u and v the components of the displacement in the radial and tangential directions, respectively. If u is the radial displacement of the side ad

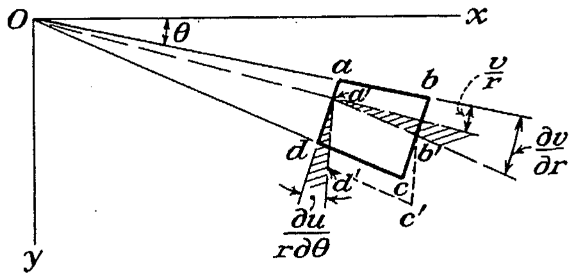

Fig. 44

of the element *abcd* (Fig. 44), the radial displacement of the side *bc* is $u + (\partial u/\partial r)\, dr$. The unit elongation of the element *abcd* in the radial direction is then

$$\epsilon_r = \frac{\partial u}{\partial r} \tag{48}$$

The strain in the tangential direction depends not only on the displacement v but also on the radial displacement u. Assuming, for instance, that the points a and d of the element *abcd* (Fig. 44) have only the radial displacement u, the new length of the arc *ad* is $(r + u)\, d\theta$ and the tangential strain is therefore

$$\frac{(r + u)\, d\theta - r\, d\theta}{r\, d\theta} = \frac{u}{r}$$

The difference in the tangential displacement of the sides *ab* and *cd* of the element *abcd* is $(\partial v/\partial\theta)\, d\theta$, and the tangential strain due to the displacement v is accordingly $\partial v/r\, \partial\theta$. The total tangential strain is thus[1]

$$\epsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\, \partial\theta} \tag{49}$$

Considering now the shearing strain, let $a'b'c'd'$ be the position of the element *abcd* after deformation (Fig. 44). The angle between the direction *ad* and $a'd'$ is due to the radial displacement u and is equal to $\partial u/r\, \partial\theta$. In the same manner, the angle between $a'b'$ and *ab* is equal to $\partial v/\partial r$. It should be noted that only part of this angle (shaded in the figure) contributes to the shearing strain and the other part, equal to v/r, represents the angular displacement due to rotation of the element *abcd* as a rigid body about the axis through O. Hence the total change in the angle *dab*, representing the shearing strain, is

$$\gamma_{r\theta} = \frac{\partial u}{r\, \partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{50}$$

[1] The symbol ϵ_θ was used with a different meaning in Art. 11.

Substituting now the expressions for the strain components (48), (49), (50) into the equations of Hooke's law for plane stress,

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \tag{51}$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

we can obtain sufficient equations for determining u and v.

31 | Displacements for Symmetrical Stress Distributions

Substituting in the first of Eqs. (51) the stress components from Eqs. 42, we find

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left[\frac{(1+\nu)A}{r^2} + 2(1-\nu)B\log r + (1-3\nu)B + 2(1-\nu)C\right]$$

By integration we obtain

$$u = \frac{1}{E}\left[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\log r - B(1+\nu)r + 2C(1-\nu)r\right] + f(\theta) \tag{a}$$

in which $f(\theta)$ is a function of θ only. From the second of Eqs. (51), we find, by using Eq. 49,

$$\frac{\partial v}{\partial \theta} = \frac{4Br}{E} - f(\theta)$$

from which, by integration

$$v = \frac{4Br\theta}{E} - \int f(\theta)\,d\theta + f_1(r) \tag{b}$$

where $f_1(r)$ is a function of r only. Substituting (a) and (b) in Eq. (50) and ~~noting~~ that $\gamma_{r\theta}$ is zero since $\tau_{r\theta}$ is zero, we find

$$\frac{1}{r}\frac{\partial f(\theta)}{\partial \theta} + \frac{\partial f_1(r)}{\partial r} + \frac{1}{r}\int f(\theta)\,d\theta - \frac{1}{r}f_1(r) = 0 \tag{c}$$

from which

$$f_1(r) = Fr \qquad f(\theta) = H\sin\theta + K\cos\theta \tag{d}$$

where F, H, and K are constants to be determined from the conditions of constraint of the curved bar or ring. Substituting expressions (d) into

Eqs. (*a*) and (*b*), we find the following expressions for the displacements:[1]

$$u = \frac{1}{E}\left[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\log r - B(1+\nu)r + 2C(1-\nu)r\right] + H\sin\theta + K\cos\theta \tag{52}$$

$$v = \frac{4Br\theta}{E} + Fr + H\cos\theta - K\sin\theta$$

in which the values of constants A, B, and C for each particular case should be substituted. Consider, for instance, pure bending. Taking the centroid of the cross section from which θ is measured (Fig. 42) and also an element of the radius at this point, as rigidly fixed, the conditions of constraint are

$$u = 0 \qquad v = 0 \qquad \frac{\partial v}{\partial r} = 0 \qquad \text{for } \theta = 0 \text{ and } r = r_0 = \frac{a+b}{2}$$

Applying these to expressions (52), we obtain the following equations for calculating the constants of integration F, H, and K:

$$\frac{1}{E}\left[-\frac{(1+\nu)A}{r_0} + 2(1-\nu)Br_0\log r_0 - B(1+\nu)r_0 + 2C(1-\nu)r_0\right] + K = 0$$

$$Fr_0 + H = 0$$

$$F = 0$$

From this it follows that $F = H = 0$, and for the displacement v we obtain

$$v = \frac{4Br\theta}{E} - K\sin\theta \tag{53}$$

This means that the displacement of any cross section consists of a translatory displacement $-K\sin\theta$, the same for all points in the cross section, and of a rotation of the cross section by the angle $4B\theta/E$ about the center of curvature O (Fig. 42). We see that cross sections remain plane in pure bending as is usually assumed in the elementary theory of the bending of curved bars.

In discussing the symmetrical stress distribution in a full ring (page 69) the constant B in the general solution (42) was taken as zero, and in this manner we arrived at a solution of Lamé's problem. Now, after obtaining expressions (52) for displacements, we see what is implied by taking B as zero. B contributes to the displacement v the term $4Br\theta/E$. This term is not *single-valued;* it changes when we increase θ by 2π, that is,

[1] Equation (*c*) is satisfied only when $\int f(\theta)\,d\theta$ is taken from (*d*) without an additive constant.

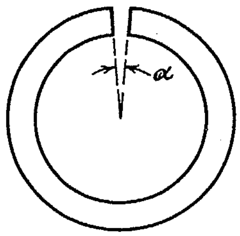

Fig. 45

if we return to a given point after making a complete circle round the ring. Such a *many-valued* expression for a displacement is physically impossible in a full ring, and so, for this case, we must take $B = 0$ in the general solution (42).

A full ring is an example of a multiply-connected body, that is, a body such that some sections can be cut clear across without dividing the body into two parts. In determining the stresses in such bodies, the boundary conditions referring to the stresses are not sufficient to determine completely the stress distribution, and additional equations, representing the conditions that the displacements should be single-valued, must be considered (see Arts. 34 and 43).

The physical meaning of many-valued solutions can be explained by considering *initial stresses* in a multiply-connected body. If a portion of the ring between two adjacent cross sections is cut out (Fig. 45) and the ends of the ring are joined again by welding or other means, a ring with initial stresses is obtained, that is, there are stresses in the ring when external forces are absent. If α is the small angle measuring the portion of the ring that was cut out, the tangential displacement necessary to bring the ends of the ring together is

$$v = \alpha r \tag{e}$$

The same displacement, obtained from Eq. (53) by putting $\theta = 2\pi$, is

$$v = 2\pi \frac{4Br}{E} \tag{f}$$

From (*e*) and (*f*) we find

$$B = \frac{\alpha E}{8\pi} \tag{g}$$

The constant B, entering into the many-valued term for the displacement (53), has now a definite value depending on the way in which the initial stresses were produced in the ring. Substituting (*g*) into Eqs. (*f*) of Art. 29, we find that the bending moment necessary to bring the ends of the ring together (Fig. 45) is

$$M = -\frac{\alpha E}{8\pi} \frac{(b^2 - a^2)^2 - 4a^2b^2[\log (b/a)]^2}{2(b^2 - a^2)} \tag{h}$$

The initial stresses in the ring can easily be calculated from this by using the solution (47) for pure bending.

32 | Rotating Disks

The stress distribution in rotating circular disks is of great practical importance.[1] If the thickness of the disk is small in comparison with its radius, the variation of radial and tangential stresses over the thickness can be neglected[2] and the problem can be easily solved.[3] If the thickness of the disk is constant Eqs. (37) can be applied, and it is only necessary to put the body force equal to the inertia force.[4] Then

$$R = \rho\omega^2 r \qquad S = 0 \tag{a}$$

where ρ is the mass per unit volume of the material of the disk and ω the angular velocity of the disk. Because of the symmetry, $\tau_{r\theta}$ vanishes and σ_r, σ_θ are independent of θ. The second of Eqs. (37) is identically satisfied. The first can be written in the form

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0 \tag{b}$$

The strain components in the case of symmetry are, from Eqs. (48) and (49)

$$\epsilon_r = \frac{du}{dr} \qquad \epsilon_\theta = \frac{u}{r} \tag{c}$$

We can solve the first two stress-strain relations (51) as equations for the stress components to obtain

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \qquad \sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r)$$

and then, using (c),

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \qquad \sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \tag{d}$$

[1] A complete discussion of this problem, and an extensive bibliography of the subject can be found in the book K. Löffler, "Die Berechnung von Rotierenden Scheiben und Schalen," Springer-Verlag OHG, Göttingen, Germany, 1961.

[2] An exact solution of the problem for a disk having the shape of a flat ellipsoid of revolution was obtained by C. Chree, see *Proc. Roy. Soc. (London)*, vol. 58, p. 39, 1895. It shows that the difference between the maximum and the minimum stress at the axis of revolution is only 5 percent of the maximum stress in a uniform disk with thickness one-eighth of its diameter.

[3] A more detailed discussion of the problem will be given later (see Art. 134).

[4] The weight of the disk is neglected.

By taking the stress function in the form

$$\phi = f(r) \cos \theta$$

and proceeding as above, we get a solution for the case when a vertical force and a couple are applied to the upper end of the bar (Fig. 46). Subtracting from this solution the stresses produced by the couple (see Art. 29), the stresses due to a vertical force applied at the upper end of the bar remain. Having the solutions for a horizontal and for a vertical load, the solution for any inclined force can be obtained by superposition.

In the above discussion it was always assumed that Eqs. (*e*) are satisfied and that the circular boundaries of the bar are free from forces. By taking the expressions in (*e*) different from zero, we obtain the case when normal and tangential forces proportional to $\sin \theta$ and $\cos \theta$ are distributed over circular boundaries of the bar. Combining such solutions with the solutions previously obtained for pure bending and for bending by a force applied at the end we can approach the condition of loading of a vault covered with sand or soil.[1]

34 | Edge Dislocation

In Art. 33 the displacement components (*q*) were derived from the stress-components (59). The constants A, B, D were given by (*g*) for the problem illustrated in Fig. 46.

The application of this solution to the quarter ring was a matter of choice, not necessity. The same solution can be applied to a nearly complete ring, Fig. 48*a* or *b*. We can also interpret it for imposed *displacements* instead of imposed forces.

Considering the displacements (*q*) of Art. 33, we observe that the first term in the expression for u can give rise to a *discontinuity*. In Fig. 48*b*, a fine radial saw cut has been made in the originally complete ring, at $\theta = 0$. The lower face of the cut has $\theta = 0$. The upper face has $\theta = 2\pi - \epsilon$, where ϵ is infinitesimal. If u in (*q*) is evaluated for

[1] Several examples of this kind were discussed by Golovin, *loc. cit.*, and Ribière, *loc. cit.*

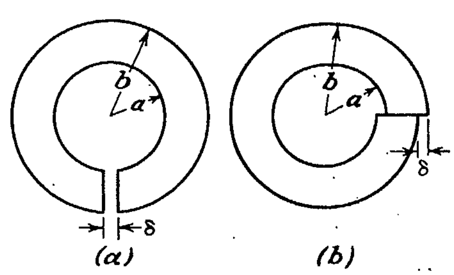

Fig. 48

these two values of θ, the results differ by an amount δ. Thus

$$\delta = (u)_{\theta=2\pi-\epsilon} - (u)_{\theta=0} \tag{a}$$

From (q) we then have

$$\delta = -\frac{2D}{E}2\pi \tag{b}$$

This relative displacement of the two faces of the cut is indicated by δ in Fig. 48b. The forces P required to effect it are found from the last of (g), Art. 33, with D given by (b) above. If the two faces are welded together after the relative displacement δ has been imposed, each applies the required force P to the other, as action and reaction. The ring is in a self-strained state, called an "edge dislocation." The corresponding plane strain state is the basis of the explanation of plastic deformation in metal crystals.[1]

Figure 48a shows a ring with a parallel gap of thickness δ. If a thin saw cut is first made, then relative displacements are imposed to open up the gap, the discontinuity of displacement now occurs in v, not u. It can be obtained from the solution of Art. 33 by taking the right-hand face of the cut at $\theta = -\pi/2$, the left-hand face at $\theta = 3\pi/2$. We then have (since v is in the direction θ-increasing)

$$\delta = (v)_{\theta=-\pi/2} - (v)_{\theta=3\pi/2} \tag{c}$$

Using the second of Eqs. (q) in Art. 33 we now find

$$\delta = \frac{2D}{E}\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) - \frac{2D}{E}\frac{3\pi}{2}\sin\frac{3\pi}{2} = \frac{4\pi D}{E} \tag{d}$$

The fact that the values of δ in (b) and (d) differ only in sign means that the stresses in the two cases will also differ only in sign. P is found from the third of (g) in Art. 33, then A and B follow from the first two. This correspondence is predictable from the fact that if the cuts of Fig. 48a and b are both made, the quadrant is cut free. The relative displacement δ of Fig. 48a, and a relative displacement $-\delta$ in Fig. 48b, can be effected simultaneously by sliding the quadrant to the right by the amount δ. No stress is induced by this, and therefore the two dislocations must have equal and opposite stresses when existing separately. This is an instance of a general[2] "theorem of equivalent cuts."

[1] G. I. Taylor, *Proc. Roy. Soc.* (London), ser. A, vol. 134, pp. 362–387, 1934. Or see, for instance, A. H. Cottrell, "Dislocations and Plastic Flow in Crystals," chap. 2, 1956.

[2] The demonstration used here was given by J. N. Goodier, *Proc. 5th Intern. Congr. Appl. Mech.*, pp. 129–133, 1938. The theorem is due to V. Volterra, who gave a general theory in *Ann. Ecole. Norm. (Paris)*, ser. 3, vol. 24, pp. 401–517, 1907. See also A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 221, Cambridge University Press, New York, 1927; A. Timpe, *Z. Math. Physik, loc. cit.*

chapter 10

Torsion

104 | Torsion of Straight Bars

It has already been shown (Art. 101) that the exact solution of the torsional problem for a circular shaft is obtained if we assume that the cross sections of the bar remain plane and rotate without any distortion during twist. This theory, developed by Coulomb,[1] was applied later by Navier[2] to bars of noncircular cross sections. Making the above assumption he arrived at the erroneous conclusions that, for a given torque, the angle of twist of bars is inversely proportional to the centroidal polar moment of inertia of the cross section, and that the maximum shearing stress occurs at the points most remote from the centroid of the cross section.[3] It is easy to see that the above assumption is in contradiction with the boundary conditions. Take, for instance, a bar of rectangular cross section (Fig. 149). From Navier's assumption it follows that at any point A on the boundary the shearing stress should act in the direction perpen-

[1] "Histoire de l'Académie," 1784, pp. 229–269, Paris, 1787.

[2] Navier, "Résumé des Leçons sur l'Application de la Mécanique," 3d ed., Paris, 1864, edited by Saint-Venant.

[3] These conclusions are correct for a thin elastic layer, corresponding to a slice of the bar between two cross sections, attached to rigid plates. See J. N. Goodier, *J. Appl. Phys.*, vol. 13, p. 167, 1942.

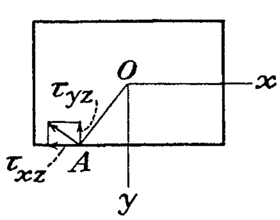

Fig. 149

dicular to the radius OA. Resolving this stress into two components τ_{xz} and τ_{yz}, it is evident that there should be a complementary shearing stress, equal to τ_{yz}, on the element of the lateral surface of the bar at the point A (see page 5), which is in contradiction with the assumption that the lateral surface of the bar is free from external forces, the twist being produced by couples applied at the ends. A simple experiment with a rectangular bar, represented in Fig. 150, shows that the cross sections of the bar do not remain plane during torsion, and that the distortions of rectangular elements on the surface of the bar are greatest at the middles of the sides, i.e., at the points which are nearest to the axis of the bar.

The correct solution of the problem of torsion of bars by couples applied at the ends was given by Saint-Venant.[1]

[1] "Mémoires Savants Etrangers," vol. 14, 1855. See also Saint-Venant's note to Navier's book, *loc. cit.*, and I. Todhunter and K. Pearson, "History of the Theory of Elasticity," vol. 2.

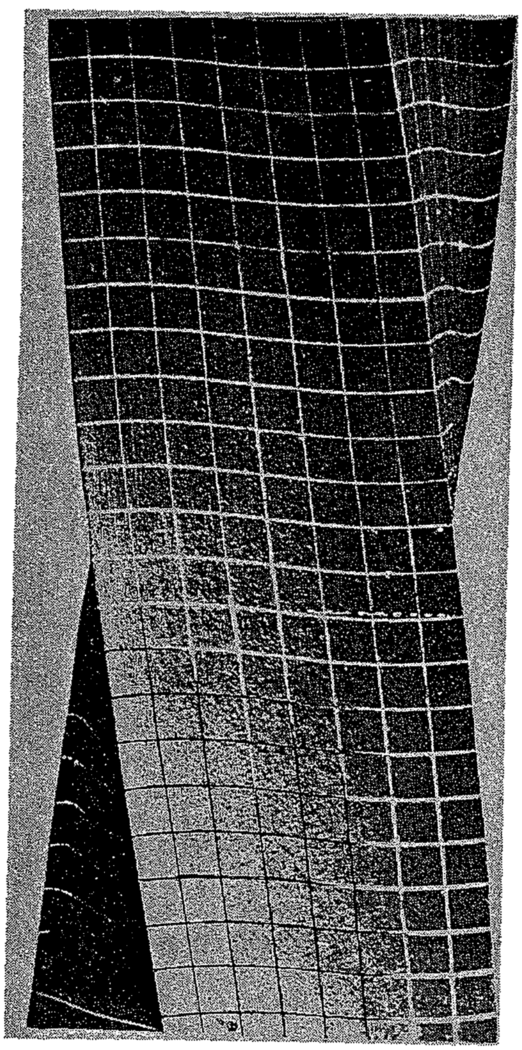

Fig. 150

He used the so-called *semi-inverse method*. That is, at the start he made certain assumptions as to the deformation of the twisted bar and showed that with these assumptions he could satisfy the equations of equilibrium (123) and the boundary conditions (124). Then from the uniqueness of solutions of the elasticity equations (Art. 96) it follows that the assumptions made at the start are correct and the solution obtained is the exact solution of the torsion problem, provided that the torques on the ends are applied as shear stress in exactly the manner required by the solution itself.

Consider a uniform bar of any cross section twisted by couples applied at the ends, Fig. 151. Guided by the solution for a circular shaft (page 282), Saint-Venant assumes that the deformation of the twisted shaft consists (1) of rotations of cross sections of the shaft as in the case of a circular shaft and (2) of *warping* of the cross sections which is the same for all cross sections. Taking the origin of coordinates in an end cross section (Fig. 151), we find that the displacements corresponding to rotation of cross sections are

$$u = -\theta z y \qquad v = \theta z x \tag{a}$$

where θz is the angle of rotation of the cross section at a distance z from the origin.

The warping of cross sections is defined by a function ψ by writing

$$w = \theta\psi(x,y) \tag{b}$$

With the assumed[1] displacements (a) and (b) we calculate the components of strain from Eqs. (2), which give

$$\begin{aligned}
\epsilon_x &= \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta\left(\frac{\partial \psi}{\partial x} - y\right) \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta\left(\frac{\partial \psi}{\partial y} + x\right)
\end{aligned} \tag{c}$$

[1] It has been shown that no other form of displacement linear in the twist θ could exist if every thin slice of the bar is in the same state. See J. N. Goodier and W. S. Shaw, *J. Mech. Phys. Solids*, vol. 10, pp. 35–52, 1962.

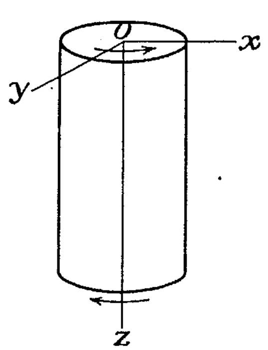

Fig. 151

The corresponding components of stress, from Eqs. (3) and (6), are

$$
\begin{aligned}
\sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\
\tau_{xz} &= G\theta\left(\frac{\partial\psi}{\partial x} - y\right) \\
\tau_{yz} &= G\theta\left(\frac{\partial\psi}{\partial y} + x\right)
\end{aligned} \tag{d}
$$

It can be seen that with the assumptions (*a*) and (*b*) regarding the deformation, there will be no normal stresses acting between the longitudinal fibers of the shaft or in the longitudinal direction of those fibers. There also will be no distortion in the planes of cross sections, since ϵ_x, ϵ_y, γ_{xy} vanish. We have at each point pure shear, defined by the components τ_{xz} and τ_{yz}. The function $\psi(x,y)$, defining warping of cross section, must now be determined in such a way that equations of equilibrium (123) will be satisfied. Substituting expressions (*d*) in these equations and neglecting body forces, we find that the function ψ must satisfy the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{147}$$

Consider now the boundary conditions (124). For the lateral surface of the bar, which is free from external forces and has normals perpendicular to the z axis, we have $\bar{X} = \bar{Y} = \bar{Z} = 0$ and $\cos Nz = n = 0$. The first two of Eqs. (124) are identically satisfied and the third gives

$$\tau_{xz}l + \tau_{yz}m = 0 \tag{e}$$

which means that the resultant shearing stress at the boundary is directed along the tangent to the boundary, Fig. 152. It was shown before (see page 292) that this condition must be satisfied if the lateral surface of the bar is free from external forces.

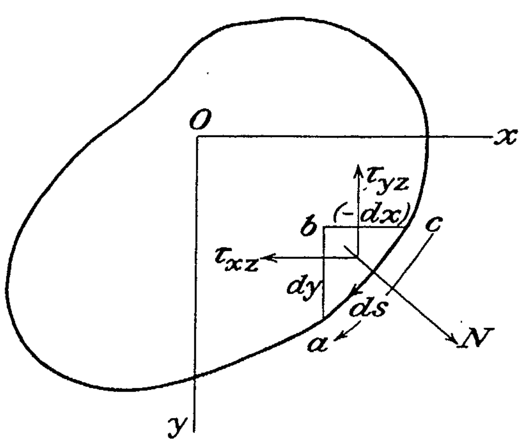

Fig. 152

Considering an infinitesimal element abc at the boundary and assuming that s is increasing in the direction from c to a, we have

$$l = \cos Nx = \frac{dy}{ds} \qquad m = \cos Ny = -\frac{dx}{ds}$$

and Eq. (e) becomes

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \tag{148}$$

Thus each problem of torsion is reduced to the problem of finding a function ψ satisfying Eq. (147) and the boundary condition (148).

An alternative procedure, which has the advantage of leading to a simpler boundary condition, is as follows. In view of the vanishing of σ_x, σ_y, σ_z, τ_{xy} [Eqs. (d)], the equations of equilibrium (123) reduce to

$$\frac{\partial\tau_{xz}}{\partial z} = 0 \qquad \frac{\partial\tau_{yz}}{\partial z} = 0 \qquad \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0$$

The first two are already satisfied since τ_{xz} and τ_{yz}, as given by Eqs. (d), are independent of z. The third means that we can express τ_{xz} and τ_{yz} as

$$\tau_{xz} = \frac{\partial\phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial\phi}{\partial x} \tag{149}$$

where ϕ is a function of x and y, called the *stress function*.[1]

From Eqs. (149) and (d) we have

$$\frac{\partial\phi}{\partial y} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right) \qquad -\frac{\partial\phi}{\partial x} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right) \tag{f}$$

Eliminating ψ by differentiating the first with respect to y, the second with respect to x, and subtracting from the first, we find that the stress function must satisfy the differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = F \tag{150}$$

where

$$F = -2G\theta \tag{151}$$

The boundary condition (e) becomes, introducing Eqs. (149),

$$\frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} = 0 \tag{152}$$

This shows that the stress function ϕ must be constant along the boundary of the cross section. In the case of singly connected sections, e.g., for solid bars, this constant can be chosen arbitrarily, and in the following discussion we shall take it equal to zero. Thus the determina-

[1] It was introduced by L. Prandtl. See *Physik. Z.*, vol. 4, 1903.

tion of the stress distribution over a cross section of a twisted bar consists in finding the function ϕ that satisfies Eq. (150) and is zero at the boundary. Several applications of this general theory to particular shapes of cross sections will be shown later.

Let us consider now the conditions at the ends of the twisted bar. The normals to the end cross sections are parallel to the z axis. Hence $l = m = 0$, $n = \pm 1$ and Eqs. (124) become

$$\bar{X} = \pm \tau_{xz} \qquad \bar{Y} = \pm \tau_{yz} \tag{g}$$

in which the + sign should be taken for the end of the bar for which the external normal has the direction of the positive z axis, as for the lower end of the bar in Fig. 151. We see that over the ends the shearing forces are distributed in the same manner as the shearing stresses over the cross sections of the bar. It is easy to prove that these forces give us a torque. Substituting in Eqs. (g) from (149) and observing that ϕ at the boundary is zero, we find

$$\iint \bar{X}\, dx\, dy = \iint \tau_{xz}\, dx\, dy = \iint \frac{\partial \phi}{\partial y}\, dx\, dy = \int dx \int \frac{\partial \phi}{\partial y}\, dy = 0$$

$$\iint \bar{Y}\, dx\, dy = \iint \tau_{yz}\, dx\, dy = -\iint \frac{\partial \phi}{\partial x}\, dx\, dy = -\int dy \int \frac{\partial \phi}{\partial x}\, dx = 0$$

Thus the resultant of the forces distributed over the ends of the bar is zero, and these forces represent a couple the magnitude of which is

$$M_t = \iint (\bar{Y}x - \bar{X}y)\, dx\, dy = -\iint \frac{\partial \phi}{\partial x} x\, dx\, dy - \iint \frac{\partial \phi}{\partial y} y\, dx\, dy \tag{h}$$

Integrating this by parts, and observing that $\phi = 0$ at the boundary, we find

$$M_t = 2\iint \phi\, dx\, dy \tag{153}$$

each of the integrals in the last member of Eqs. (h) contributing one half of this torque. Thus we find that half the torque is due to the stress component τ_{xz} and the other half to τ_{yz}.

We see that by assuming the displacements (a) and (b), and determining the stress components τ_{xz}, τ_{yz} from Eqs. (149), (150), and (152), we obtain a stress distribution that satisfies the equations of equilibrium (123), leaves the lateral surface of the bar free from external forces, and sets up at the ends the torque given by Eq. (153). The compatibility conditions (126) need not be considered. The stress has been derived from the displacements (a) and (b). The question of compatibility reduces to the existence of the single displacement function ψ, which is ensured by (150) as the result of eliminating ψ from (f). Thus all the equations of elas-

ticity are satisfied and the solution obtained in this manner is the exact solution of the torsion problem.

It was pointed out that the solution requires that the forces at the ends of the bar should be distributed in a definite manner. But the practical application of the solution is not limited to such cases. From Saint-Venant's principle it follows that in a long twisted bar, at a sufficient distance from the ends, the stresses depend only on the magnitude of the torque M_t and are practically independent of the manner in which the tractions are distributed over the ends.

105 | Elliptic Cross Section

Let the boundary of the cross section (Fig. 153) be given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{a}$$

Then Eq. (150) and the boundary condition (152) are satisfied by taking the stress function in the form

$$\phi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{b}$$

in which m is a constant. Substituting (b) into Eq. (150), we find

$$m = \frac{a^2b^2}{2(a^2 + b^2)} F$$

Hence

$$\phi = \frac{a^2b^2F}{2(a^2 + b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{c}$$

The magnitude of the constant F will now be determined from Eq. (153). Substituting in this equation from (c), we find

$$M_t = \frac{a^2b^2F}{a^2 + b^2}\left(\frac{1}{a^2}\iint x^2\,dx\,dy + \frac{1}{b^2}\iint y^2\,dx\,dy - \iint dx\,dy\right) \tag{d}$$

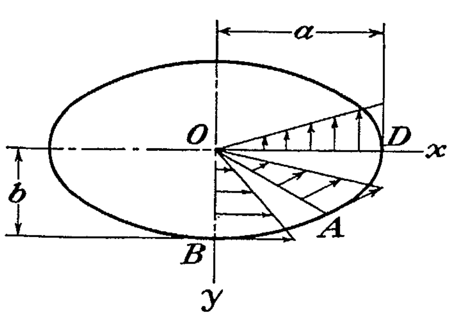

Fig. 153

Since

$$\iint x^2\,dx\,dy = I_y = \frac{\pi ba^3}{4} \qquad \iint y^2\,dx\,dy = I_x = \frac{\pi ab^3}{4} \qquad \iint dx\,dy = \pi ab$$

we find, from (d),

$$M_t = -\frac{\pi a^3b^3F}{2(a^2+b^2)}$$

from which

$$F = -\frac{2M_t(a^2+b^2)}{\pi a^3b^3} \tag{e}$$

Then, from (c),

$$\phi = -\frac{M_t}{\pi ab}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \tag{f}$$

Substituting in Eqs. (149), the stress components are

$$\tau_{xz} = -\frac{2M_ty}{\pi ab^3} \qquad \tau_{yz} = \frac{2M_tx}{\pi a^3b} \tag{154}$$

The ratio of the stress components is proportional to the ratio y/x and hence is constant along any radius such as OA (Fig. 153). This means that the resultant shearing stress along any radius OA has a constant direction that evidently coincides with the direction of the tangent to the boundary at the point A. Along the vertical axis OB the stress component τ_{yz} is zero, and the resultant stress is equal to τ_{xz}. Along the horizontal axis OD the resultant shearing stress is equal to τ_{yz}. It is evident that the maximum stress is at the boundary, and it can easily be proved that this maximum occurs at the ends of the minor axis of the ellipse. Substituting $y = b$ in the first of Eqs. (154), we find that the absolute value of this maximum is

$$\tau_{\max} = \frac{2M_t}{\pi ab^2} \tag{155}$$

For $a = b$ this formula coincides with the well-known formula for a circular cross section.

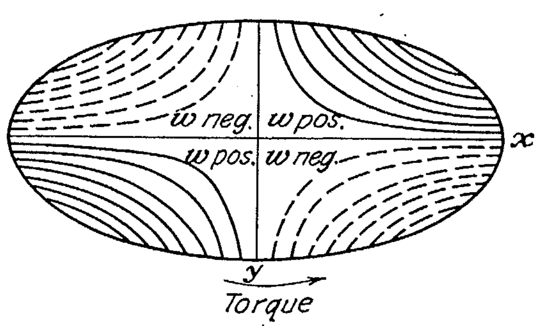

Fig. 154

Substituting (*e*) in Eq. (151) we find the expression for the angle of twist

$$\theta = M_t \frac{a^2 + b^2}{\pi a^3 b^3 G} \tag{156}$$

The factor by which we divide torque to obtain the twist per unit length is called the *torsional rigidity*. Denoting it by C, its value for the elliptic cross section, from (156), is

$$C = \frac{\pi a^3 b^3 G}{a^2 + b^2} = \frac{G}{4\pi^2} \frac{(A)^4}{I_p} \tag{157}$$

in which

$$A = \pi ab \qquad I_p = \frac{\pi ab^3}{4} + \frac{\pi ba^3}{4}$$

are the area and centroidal moment of inertia of the cross section.

Having the stress components (154) we can easily obtain the displacements. The components u and v are given by Eqs. (*a*) of Art. 104. The displacement w is found from Eqs. (*d*) and (*b*) of Art. 104. Substituting from Eqs. (154) and (156) and integrating, we find

$$w = M_t \frac{(b^2 - a^2)xy}{\pi a^3 b^3 G} \tag{158}$$

This shows that the contour lines for the warped cross section are hyperbolas having the principal axes of the ellipse as asymptotes (Fig. 154).

106 | Other Elementary Solutions

In studying the torsional problem, Saint-Venant discussed several solutions of Eq. (150) in the form of polynomials. To solve the problem, let us represent the stress function in the form

$$\phi = \phi_1 + \frac{F}{4}(x^2 + y^2) \tag{a}$$

Then, from Eq. (150),

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \tag{b}$$

and along the boundary, from Eq. (152),

$$\phi_1 + \frac{F}{4}(x^2 + y^2) = \text{const} \tag{c}$$

Thus the torsional problem is reduced to obtaining solutions of Eq. (*b*) satisfying the boundary condition (*c*). To get solutions in the form of polynomials, we take the function of the complex variable

$$(x + iy)^n \tag{d}$$

The real and the imaginary parts of this expression are each solutions of Eq. (*b*) (see page 171). Taking, for instance, $n = 2$ we obtain the solutions $x^2 - y^2$ and $2xy$. With $n = 3$ we obtain solutions $x^3 - 3xy^2$ and $3x^2y - y^3$. With $n = 4$, we arrive

in which the integration must be extended over the length of the ring. Denoting by A the area bounded by the ring and observing that τ is the slope, so that $\tau\delta$ is the difference in level h of the two adjacent contour lines, we find, from (f),

$$dM_t = 2hA \tag{g}$$

i.e., the torque corresponding to the elemental ring is given by twice the volume shaded in the figure. The total torque is given by the sum of these volumes, i.e., twice the volume between AB, the membrane AC and DB, and the flat plate CD. The conclusion follows similarly for several holes.

116 | Torsion of Thin Tubes

An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 172) represent the levels of the outer and the inner boundaries, and AC and DB be the cross section of the membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that AC and BD are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \tag{a}$$

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t, we apply again the membrane analogy and calculate the torque from the volume

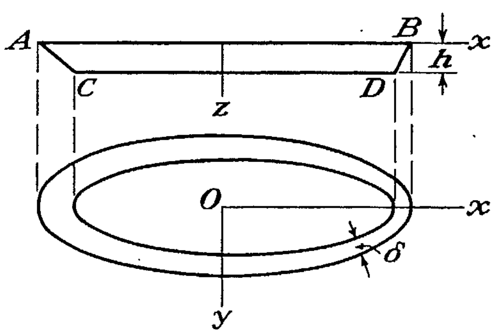

Fig. 172

$ACDB$. Then

$$M_t = 2Ah = 2A\delta\tau \tag{b}$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses:

$$\tau = \frac{M_t}{2A\delta} \tag{176}$$

For determining the angle of twist θ, we apply Eq. (160). Then

$$\tau\, ds = \frac{M_t}{2A}\int \frac{ds}{\delta} = 2G\theta A \tag{c}$$

from which[1]

$$\theta = \frac{M_t}{4A^2G}\int \frac{ds}{\delta} \tag{177}$$

In the case of a tube of uniform thickness, δ is constant and (177) gives

$$\theta = \frac{M_t s}{4A^2G\delta} \tag{178}$$

in which s is the length of the centerline of the ring section of the tube.

If the tube has reentrant corners, as in the case represented in Fig. 173, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (176) and depends on the radius a of the fillet of the reentrant corner (Fig. 173b). In calculating this maximum stress, we shall use the membrane analogy as we

[1] Equations (176) and (177) for thin tubular sections were obtained by R. Bredt, *VDI*, vol. 40, p. 815, 1896.

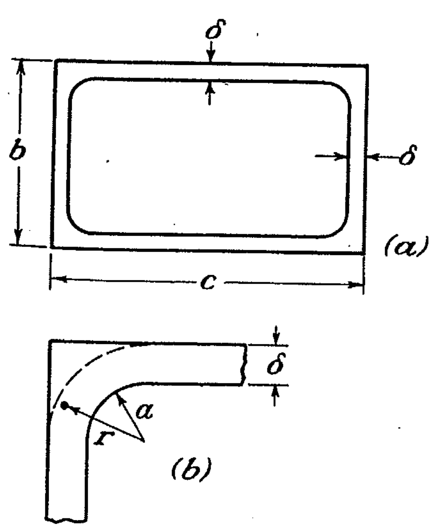

Fig. 173

did for the reentrant corners of rolled sections (Art. 112). The equation of the membrane at the reentrant corner may be taken in the form

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} = -\frac{q}{S}$$

Replacing q/S by $2G\theta$ and noting that $\tau = -dz/dr$ (see Fig. 172), we find

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = 2G\theta \tag{d}$$

Assuming that we have a tube of constant thickness δ and denoting by τ_0 the stress at a considerable distance from the corner calculated from Eq. (176), we find, from (c),

$$2G\theta = \frac{\tau_0 s}{A}$$

Substituting in (d),

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = \frac{\tau_0 s}{A} \tag{e}$$

The general solution of this equation is

$$\tau = \frac{C}{r} + \frac{\tau_0 sr}{2A} \tag{f}$$

Assuming that the projecting angles of the cross section have fillets with the radius a, as indicated in the figure, the constant of integration C can be determined from the equation

$$\int_a^{a+\delta} \tau\, dr = \tau_0 \delta \tag{g}$$

which follows from the hydrodynamical analogy (Art. 114), viz.: if an ideal fluid circulates in a channel having the shape of the ring cross section of the tubular member, the quantity of fluid passing each cross section of the channel must remain constant. Substituting expression (f) for τ into Eq. (g), and integrating, we find that

$$C = \tau_0 \delta \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)}$$

and, from Eq. (f), that

$$\tau = \frac{\tau_0 \delta}{r} \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)} + \frac{\tau_0 sr}{2A} \tag{h}$$

For a thin-walled tube the ratios $s(2a + \delta)/A$, sr/A, will be small, and (h) reduces to

$$\tau = \frac{\tau_0 \delta / r}{\log_e (1 + \delta/a)} \tag{i}$$

Substituting $r = a$ we obtain the stress at the reentrant corner. This is plotted in Fig. 174. The other curve[1] (A in Fig. 174) was obtained by the method of finite differences, without the assumption that the membrane at the corner has the form of a surface of revolution. It confirms the accuracy of Eq. (i) for small fillets—say up to $a/\delta = \frac{1}{4}$. For larger fillets the values given by Eq. (i) are too high.

Let us consider now the case when the cross section of a tubular member has more than two boundaries. Taking, for example, the case shown in Fig. 175 and assuming that the thickness of the wall is very small, the shearing stresses in each portion of the wall, from the membrane analogy, are

$$\tau_1 = \frac{h_1}{\delta_1} \qquad \tau_2 = \frac{h_2}{\delta_2} \qquad \tau_3 = \frac{h_1 - h_2}{\delta_3} = \frac{\tau_1\delta_1 - \tau_2\delta_2}{\delta_3} \tag{j}$$

in which h_1 and h_2 are the levels of the inner boundaries CD and EF.[2]

The magnitude of the torque, determined by the volume $ACDEFB$, is

$$M_t = 2(A_1h_1 + A_2h_2) = 2A_1\delta_1\tau_1 + 2A_2\delta_2\tau_2 \tag{k}$$

where A_1 and A_2 are areas indicated in the figure by dotted lines.

Further equations for the solution of the problem are obtained by applying Eq. (160) to the closed curves indicated in the figure by dotted lines. Assuming that the thicknesses δ_1, δ_2, δ_3 are constant and denoting

[1] Huth, *op. cit.*

[2] It is assumed that the plates are guided so as to remain horizontal (see p. 331).

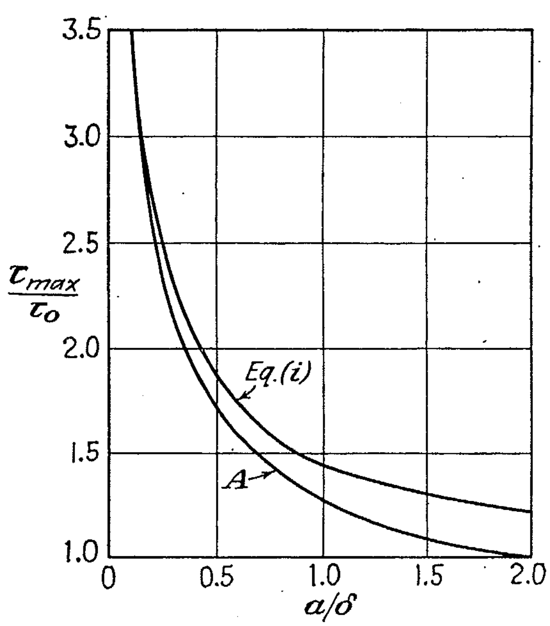

Fig. 174

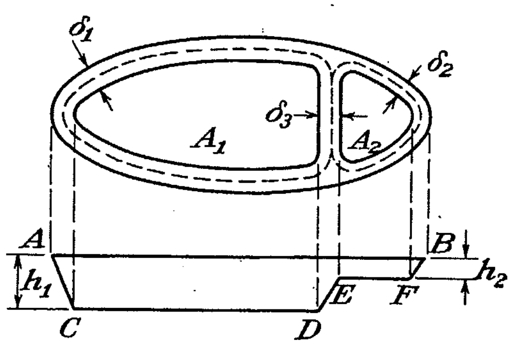

Fig. 175

by s_1, s_2, s_3 the lengths of corresponding dotted curves, we find, from Fig. 175,

$$\begin{aligned} \tau_1 s_1 + \tau_3 s_3 &= 2G\theta A_1 \\ \tau_2 s_2 - \tau_3 s_3 &= 2G\theta A_2 \end{aligned} \tag{l}$$

By using the last of the Eqs. (j) and Eqs. (k) and (l), we find the stresses τ_1, τ_2, τ_3 as functions of the torque:

$$\tau_1 = \frac{M_t[\delta_3 s_2 A_1 + \delta_2 s_3(A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \tag{m}$$

$$\tau_2 = \frac{M_t[\delta_3 s_1 A_2 + \delta_1 s_3(A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \tag{n}$$

$$\tau_3 = \frac{M_t(\delta_1 s_2 A_1 - \delta_2 s_1 A_2)}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \tag{o}$$

In the case of a symmetrical cross section, $s_1 = s_2$, $\delta_1 = \delta_2$, $A_1 = A_2$, and $\tau_3 = 0$. In this case the torque is taken by the outer wall of the tube, and the web remains unstressed.[1]

To get the twist for any section like that shown in Fig. 175, one substitutes the values of the stresses in one of the Eqs. (l). Thus θ can be obtained as a function of the torque M_t.

117 | Screw Dislocations

In the two preceding articles, we have observed the requirement that w must be a single-valued function if the solution is to represent correctly a state of torsion. On reexamining Eqs. (149), (150), and (151), and the boundary condition (152), we can quickly see that it is possible to find states of stress corresponding to $\theta = 0$. The stress function ϕ is to satisfy Laplace's equation and to be constant on each boundary curve of the sec-

[1] The small stresses corresponding to the change in slope of the membrane across the thickness of the web are neglected in this derivation.

tion. But we must use w rather than the form $\theta\psi(x,y)$ of Eq. (b) on page 293. Then Eqs. (f) of page 295 are replaced by

$$\frac{\partial \phi}{\partial y} = G\frac{\partial w}{\partial x} \qquad -\frac{\partial \phi}{\partial x} = G\frac{\partial w}{\partial y} \tag{a}$$

These are Cauchy-Riemann equations (see page 171) for the functions Gw and ϕ. Therefore, $Gw + i\phi$ is an analytic function of $x + iy$. Thus,

$$Gw + i\phi = f(x + iy) \tag{b}$$

Once the function f is chosen, we have a definite state, in which w will be the only nonzero displacement component.

Let r, ψ now represent polar coordinates in the cross section. The choice

$$f(x + iy) = -iA \log (x + iy) = A\psi - iA \log r \tag{c}$$

where A is a real constant, is of particular interest in the dislocation theory of plastic deformation (see Art. 34). From (b), we now have

$$Gw = A\psi \qquad \phi = -A \log r \tag{d}$$

The corresponding shear stress is in the circumferential direction and is given by the polar components

$$\tau_{z\psi} = -\frac{\partial \phi}{\partial r} = \frac{A}{r} \qquad \tau_{zr} = 0 \tag{e}$$

Any cylindrical boundary surface r = constant is free from loading. But the displacement w is not continuous. We can apply the solution to a hollow circular cylinder $a < r < b$ as in Fig. 176, which has an axial cut. One face is moved axially along the other by the uniform relative displacement

$$w(r,2\pi) - w(r,0) = \frac{2\pi A}{G} \tag{f}$$

obtained from the first of (d). The stress (e) can be regarded as induced

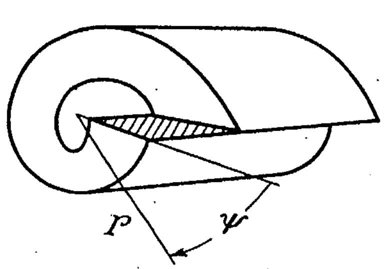

Fig. 176

by imposing this relative displacement, together with the shear loading on the ends implied by (e). This loading forms a torque

$$2\pi \int_a^b \tau_{z\psi} r^2 \, dr = \pi(b^2 - a^2)A$$

An equal and opposite torque can be introduced by superposing a state of simple torsion (Art. 101) with

$$\tau_{z\psi} = Br \qquad \tau_{zr} = 0 \qquad B = \frac{-2A}{a^2 + b^2}$$

and with $w = 0$. Finally, then, we have stress

$$\tau_{z\psi} = A\left(\frac{1}{r} - \frac{2r}{a^2 + b^2}\right) \tag{g}$$

which may be attributed to the relative displacement (f), the end torque being zero. There is still, of course, a distribution of shear stress on the ends, represented by (g). Since its resultant vanishes, its removal would have only a local effect, according to Saint-Venant's principle.

This final state, as applied in materials science, is called a *screw dislocation.*[1] A hollow cylinder with a cut has six distinct types of dislocation, in each of which the strain is continuous across the cut. The screw dislocation, the edge dislocation of Art. 34, the parallel gap dislocation of Art. 34 applied to the same cut, and the angular gap dislocation of Art. 31 (Fig. 45), account for four of the six.[2]

118 | Torsion of a Bar in Which One Cross Section Remains Plane

In discussing torsional problems, it has always been assumed that the torque is applied by means of shearing stresses distributed over the ends of a bar in a definite manner, obtained from the solution of Eq. (150) and satisfying the boundary condition (152). If the distribution of stresses at the ends is different from this, a local irregularity in stress distribution results and the solution of Eqs. (150) and (152) can be applied with satisfactory accuracy only in regions at some distance from the ends of the bar.[3]

[1] See, for instance, A. H. Cottrell, "Dislocations and Plastic Flow in Crystals," Oxford University Press, Fair Lawn, N.J., 1953.

[2] See references in n. 2, p. 89. For screw dislocations in the hollow cone and the hollow sphere, see J. N. Goodier and J. C. Wilhoit, *Quart. Appl. Math.*, vol. 13, pp. 263–269, 1955.

[3] The local irregularities at the ends of a circular cylinder have been discussed by F. Purser, *Proc. Roy. Irish Acad.*, Dublin, ser. A, vol. 26, p. 54, 1906. See also K. Wolf, *Sitzber. Akad. Wiss. Wien*, vol. 125, p. 1149, 1916; A. Timpe, *Math. Ann.*, vol. 71, p. 480, 1912; G. Horvay and J. A. Mirabel, *J. Appl. Mech.*, vol. 25, pp. 561–570, 1958; H. D. Conway and J. R. Moynihan, *ibid.*, vol. 31, pp. 346–348, 1964; M. Tanimura, *Tech. Repts. Osaka Univ.*, vol. 12, no. 497, pp. 93–104, 1962.